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# Non-asymptotic state estimation for a class of linear time-varying systems with unknown inputs

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**Abstract:** In this paper, we extend the modulating functions method to estimate the state and the unknown input of a linear time-varying system defined by a linear differential equation. We first estimate the unknown input by taking a truncated Jacobi orthogonal series expansion with unknown coefficients which can be estimated by the modulating functions method. Then, we estimate the state by using extended modulating functions and the estimated input. Both input and state estimators are given by exact integral formulae involving modulating functions and the noisy output. Hence, estimations at different instants can be non-asymptotically obtained using a sliding window of finite length. Numerical results are given to show the accuracy and the robustness of the proposed estimators against corrupting noises.

Keywords: Non-asymptotic estimation; Modulating functions method; State estimation; Linear time-varying systems; Unknown input.

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## 1. INTRODUCTION

State estimation for linear systems is an important research topic in automatic control, and of great interest for engineers. In fact, for cost and technological reasons, the state can not be always measured. Therefore, state estimators, such as state observers, are often needed. Observers usually converge asymptotically, which may not be useful in some applications. In this paper, we provide non-asymptotic and robust state estimators for a class of linear time-varying systems defined by a linear differential equation with an unknown input. Moreover, we provide also estimators for the input.

Among the methods that have been proposed recently for non-asymptotic state estimation is the algebraic method proposed by Fliess and Sira-Ramirez originally for linear identification [1]. The latter has been also extended to many applications, such as parameter estimation for noisy signals (see, e.g., [2, 3, 4, 5, 6]), and numerical differentiation (see, e.g., [7, 8, 9, 10, 11]). The main idea of this method is to apply some algebraic operations (such as differentiation and multiplication) to a linear differential equation of the analyzed signals in the Laplace operational domain. When returning into the time domain, we can obtain a sequence of integral equations of the analyzed signals multiplied by some weight functions of

the Jacobi orthogonal polynomial. Then, estimators are given by integral formulae. Thus, estimations at different instants can be obtained using a sliding window of finite length. Consequently, this method is algebraic and non-asymptotic. Moreover, it exhibits good robustness properties with respect to corrupting noises (see [12] for more theoretical details). When this algebraic method is used for numerical differentiation problem, thanks to the proposed integral formulae, it also refers to the *differentiation by integration method* well known for the Lanczos generalized derivative (see [13] p. 324). The obtained algebraic differentiators have been used to design algebraic non-asymptotic observers for linear and non-linear systems (see, e.g., [14, 15, 17, 18, 19]). In the linear case [14, 15, 16], the proposed differentiators were obtained via the differential equations which define the linear systems. Hence, they can be considered as *model-based differentiators*. The state variables have been accurately estimated without any time-delay. In the non-linear case [17, 18, 19], the used differentiators were obtained via the equations of truncated Taylor or Jacobi orthogonal series expansions of the output (see [7, 8, 9] for more details). Hence, these differentiators can be considered as *model-free differentiators*. Good state estimations have been obtained, but with a known time-delay produced by the truncated term error. Very recently,

model-free differentiators have been extended to estimate fractional order derivatives [20, 21].

Modulating functions method was introduced by Shinbrot in [22]. This method has been widely used for linear and non-linear identification of continuous-time systems (see, e.g., [23, 24]). Recently, the modulating functions method has been extended to parameter estimation of noisy sinusoidal signals [25, 4]. However, to the best of our knowledge, it has not been extended to numerical differentiation problem. Rather than working in the Laplace operational domain as the algebraic estimation method, the basic idea of the modulating functions method is to transform the linear differential equation of the analyzed signals into a sequence of integral equations of these signals multiplied by the derivatives of different modulating functions via applying integration by parts in the time domain. Consequently, the modulating functions method has similar advantages to the algebraic estimation method. Moreover, it can be considered as a generalization of the algebraic estimation method in some cases (see, e.g., [4]). Indeed, the weight function of the Jacobi orthogonal polynomial is a modulating function. However, when tackling a complex problem, such as identification of fractional order systems, we can first be inspired by the algebraic estimation method by working in the operational domain (see, e.g., [26, 27]).

Having these ideas in mind, inspired by [14, 15], we are going to extend the modulating functions method to design model-based differentiators so as to estimate the state for a class of linear time-varying systems. Moreover, inspired by the idea of designing algebraic model-free differentiators in [7, 8], we are going to also estimate the unknown input using the modulating functions method. Let us note that to the best knowledge of the authors, the estimations both of the state and the unknown input for a linear time-varying system have never been done neither by the algebraic method nor by the classical modulating function method.

This paper is organized as follows. In Section 2, we begin with the problem formulation, then we recall the generalized integration by parts formula and the properties of the Jacobi orthogonal polynomial. In Section 3, we first give a general definition for the modulating functions. Then, we use the modulating functions method to estimate the unknown input and the state. Numerical results are given in Section 4. Finally, we give some conclusions and perspectives in Section 5.

## 2. PRELIMINARY

### 2.1 Problem formulation

Let us consider a class of linear time-varying systems which can be defined by the following linear differential equation:

$$\forall t \in I, \sum_{i=0}^M a_i(t) y^{(i)}(t) = u(t), \quad (1)$$

where  $I = [0, T] \subset \mathbb{R}_+$ ,  $y$  is the output which is sufficiently smooth enough, and  $u \in \mathcal{C}(I)$  is the input which is assumed to be unknown. Moreover, we assume that  $a_i$ , for  $i = 0, \dots, M \in \mathbb{N}$ , is piecewise  $i$ -times continuously differentiable on  $I$ , and is known. Let  $y^\varpi$  be a noisy observation of  $y$  on  $I$ :

$$\forall t \in I, y^\varpi(t) = y(t) + \varpi(t), \quad (2)$$

where  $\varpi$  is an integrable noise. In this paper, we want to estimate the unknown input and the state of the system defined by (1). For this purpose, we give some useful tools in the following subsection.

### 2.2 Jacobi orthogonal polynomials

The generalized integration by parts is a crucial tool for the use of modulating functions method. We recall this result in the following lemma which can be obtained by recursively applying the classical integration by parts method.

*Lemma 1.* Let  $f \in \mathcal{C}^l(\mathbb{R})$  and  $g \in \mathcal{C}^m(\mathbb{R})$ , where  $l, m \in \mathbb{N}^*$  with  $m \leq l$ . Then, for any interval  $[a, b] \subset \mathbb{R}$ , we have:

$$\begin{aligned} \int_a^b g(t) f^{(l)}(t) dt &= (-1)^m \int_a^b g^{(m)}(t) f^{(l-m)}(t) dt \\ &+ \sum_{k=0}^{m-1} (-1)^k \left[ g^{(k)}(t) f^{(l-1-k)}(t) \right]_{t=a}^{t=b}. \end{aligned} \quad (3)$$

The  $n^{\text{th}}$  ( $n \in \mathbb{N}$ ) order shifted Jacobi orthogonal polynomial defined on  $[0, 1]$  is given as follows (see [29] p. 775):

$$P_n^{(\alpha, \beta)}(t) = \sum_{j=0}^n \binom{n+\alpha}{j} \binom{n+\beta}{n-j} (t-1)^{n-j} t^j, \quad (4)$$

where  $\alpha, \beta \in ]-1, +\infty[$ . Let  $f$  and  $g$  be two functions belonging to  $\mathcal{C}([0, 1])$ , then the scalar product  $\langle \cdot, \cdot \rangle_{\alpha, \beta}$  of these functions is defined by (see [29] p. 774):

$$\langle f(\cdot), g(\cdot) \rangle_{\alpha, \beta} = \int_0^1 w_{\alpha, \beta}(t) f(t) g(t) dt, \quad (5)$$

where  $w_{\alpha, \beta}(t) = (1-t)^\alpha t^\beta$  is the associated weight function. Thus, by denoting its associated norm by  $\|\cdot\|_{\alpha, \beta}$ , we obtain:

$$\|P_n^{(\alpha, \beta)}\|_{\alpha, \beta}^2 = \frac{\Gamma(\alpha+n+1) \Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+n+1) \Gamma(n+1) (2n+\alpha+\beta+1)}. \quad (6)$$

where  $\Gamma(\cdot)$  is the classical Gamma function (see [29] p. 255).

Finally, let us recall that if  $f \in \mathcal{C}([a, b])$  with  $[a, b] \subset \mathbb{R}$  and  $h = b - a$ , then  $f$  can be expressed by the following Jacobi orthogonal series on  $[a, b]$ :  $\forall t \in [0, 1]$ ,

$$f(a+ht) = \sum_{i=0}^{+\infty} \frac{\langle P_i^{(\alpha, \beta)}(\cdot), f(a+h\cdot) \rangle_{\alpha, \beta}}{\|P_i^{(\alpha, \beta)}\|_{\alpha, \beta}^2} P_i^{(\alpha, \beta)}(t). \quad (7)$$

## 3. MAIN RESULTS

### 3.1 Extended modulating functions

We extend the classical modulating functions by the following definition.

*Definition 1.* Let  $[a, b] \subset \mathbb{R}$ ,  $l, k \in \mathbb{N}$  with  $k \leq l$ , and  $g$  be a function satisfying the following properties:

- (P<sub>1</sub>) :  $g \in \mathcal{C}^{l+1}([a, b])$ ;
- (P<sub>2</sub>) :  $g^{(i)}(a) = 0$ , for  $i = 0, 1, \dots, l$ ;
- (P<sub>3</sub>) :  $g^{(i)}(b) = 0$ , for  $i = 0, 1, \dots, k$ ,  
and  $g^{(k+1)}(b) \neq 0$ , if  $k < l$ .

Then,  $g$  is called  $(l, k)^{th}$  order modulating function on  $[a, b]$ . If  $g$  satisfies only the properties  $(P_1)$  and  $(P_2)$  for any  $k \in \mathbb{N}$ , then  $g$  is called  $(l, -1)^{th}$  order modulating function on  $[a, b]$ . If  $l = k$ , then  $g$  is the classical  $l^{th}$  order modulating function on  $[a, b]$  (see, [28]).

According to the previous definition, we can obtain that if  $\alpha, \beta \in \mathbb{R}_+^*$ , then the weight function  $w_{\alpha, \beta}$  of the shifted Jacobi polynomials is a  $(\lfloor \alpha \rfloor, \lfloor \beta \rfloor)^{th}$  order modulating function on  $[0, 1]$ , where  $\lfloor \alpha \rfloor$  (resp.  $\lfloor \beta \rfloor$ ) refers to the largest integer smaller than  $\alpha$  (resp.  $\beta$ ).

### 3.2 Estimation of unknown input

The model-free differentiators were proposed in [7, 8, 9] by taking a truncated Jacobi series expansion which locally estimates the analyzed signal on a small sliding window. In this subsection, we estimate the unknown input in a similar way, where the coefficients in the truncated Jacobi series expansion are unknown. Then, these unknown coefficients can be estimated using the modulating functions method. Based on this idea, the following proposition is given.

**Proposition 1.** Let  $y^\varpi$  be a noisy observation of the output  $y$  of the linear time-varying system defined by (1), and  $\{f_n\}_{n=0}^W$  be a set of  $(M-1)^{th}$  order modulating functions on  $[0, 1]$ . Then, an estimate of the unknown input  $u$  in (1) can be given by:  $\forall t \in [h, T]$  with  $h \in ]0, T]$ ,

$$\forall \tau \in [0, 1], u_e(t + (\tau - 1)h) = \sum_{i=0}^N \tilde{\lambda}_{t,i} P_i^{(\alpha, \beta)}(\tau), \quad (8)$$

where  $N, W \in \mathbb{N}$  with  $N \leq W$ ,  $P_i^{(\alpha, \beta)}(\cdot)$  is given by (4) with  $\alpha, \beta \in ]-1, +\infty[$ , and  $\tilde{\lambda}_{t,i}$ , for  $i = 0, \dots, N$ , is the solution of the following linear system:

$$A_f \begin{pmatrix} \tilde{\lambda}_{t,0} \\ \vdots \\ \tilde{\lambda}_{t,N} \end{pmatrix} = B_{y^\varpi}, \quad (9)$$

with  $A_f(n+1, i+1) = \int_0^1 f_n(\tau) P_i^{(\alpha, \beta)}(\tau) d\tau$ , and  $B_{y^\varpi}(n+1) = \sum_{i=0}^M \frac{(-1)^i}{h^i} \int_0^1 F_{t,h,i,n}^{(i)}(\tau) y^\varpi(t + (\tau - 1)h) d\tau$ ,  $F_{t,h,i,n}(\tau) = f_n(\tau) a_i(t + (\tau - 1)h)$ , for  $n = 0, \dots, W$  and  $i = 0, \dots, N$ .

**Remark 1.** Since  $a_i$  is piecewise  $i$ -times continuously differentiable on  $I$ , for  $i = 0, \dots, M$ , if there is a discontinuity on the sliding integration window  $[t-h, t]$ , then the derivatives of  $F_{t,h,i,n}$  should be understood in the distribution sense [30].

**Proof.**

**Step 1.** Using a truncated Jacobi series expansion:

For any  $t \in [h, T]$ , by taking the following change of variable  $t \rightarrow (t-h) + h\tau$  in (1), we get:  $\forall \tau \in [0, 1]$ ,

$$u(t + (\tau - 1)h) = \sum_{i=0}^M a_i(t + (\tau - 1)h) y^{(i)}(t + (\tau - 1)h). \quad (10)$$

Then, we take an  $N^{th}$  order truncated Jacobi series expansion of  $u$  on  $[t-h, t]$ :  $\forall \tau \in [0, 1]$ ,

$$u_N(t + (\tau - 1)h) := \sum_{i=0}^N \lambda_{t,i} P_i^{(\alpha, \beta)}(\tau), \quad (11)$$

where  $\lambda_{t,i} = \frac{\langle P_i^{(\alpha, \beta)}(\cdot), u(t + (\tau - 1)h) \rangle_{\alpha, \beta}}{\|P_i^{(\alpha, \beta)}\|_{\alpha, \beta}^2}$ . Let us take  $u_N$  as an estimation of  $u$ , then (10) becomes:

$$\sum_{i=0}^N \tilde{\lambda}_{t,i} P_i^{(\alpha, \beta)}(\tau) = \sum_{i=0}^M a_i(t + (\tau - 1)h) y^{(i)}(t + (\tau - 1)h), \quad (12)$$

where  $\tilde{\lambda}_{t,i}$  are the estimation of the unknown coefficients  $\lambda_{t,i}$ .

**Step 2.** Estimation of the unknown coefficients  $\lambda_{t,i}$ :

Let  $\{f_n\}_{n=0}^W$  be a set of  $(M-1)^{th}$  order modulating functions on  $[0, 1]$ . Then, by multiplying both sides of (12) by  $f_n$  and integrating from 0 to 1, we get: for  $n = 0, \dots, W$ ,  $\forall t \in [h, T]$ ,

$$\sum_{i=0}^N \tilde{\lambda}_{t,i} \int_0^1 f_n(\tau) P_i^{(\alpha, \beta)}(\tau) d\tau = \sum_{i=0}^M \int_0^1 f_n(\tau) a_i(t + (\tau - 1)h) y^{(i)}(t + (\tau - 1)h) d\tau. \quad (13)$$

Then, by applying the generalized integration by parts formula given in Lemma 1, we obtain: for  $n = 0, \dots, W$ ,  $\forall t \in [h, T]$ ,

$$\sum_{i=0}^N \tilde{\lambda}_{t,i} \int_0^1 f_n(\tau) P_i^{(\alpha, \beta)}(\tau) d\tau = \sum_{i=0}^M \frac{(-1)^i}{h^i} \int_0^1 F_{t,h,i,n}^{(i)}(\tau) y(t + (\tau - 1)h) d\tau, \quad (14)$$

where  $F_{t,h,i,n}(\tau) = f_n(\tau) a_i(t + (\tau - 1)h)$ . Let us mention that all the boundary derivative values are eliminated by the properties  $(P_2)$  and  $(P_3)$  of  $f_n$ . Consequently, the coefficients,  $\tilde{\lambda}_{t,i}$ , for  $i = 0, \dots, N$ , can be calculated by solving the following linear system:

$$A_f \begin{pmatrix} \tilde{\lambda}_{t,0} \\ \vdots \\ \tilde{\lambda}_{t,N} \end{pmatrix} = B_y, \quad (15)$$

where  $A_f(n+1, i+1) = \int_0^1 f_n(\tau) P_i^{(\alpha, \beta)}(\tau) d\tau$ , and  $B_y(n+1) = \sum_{i=0}^M \frac{(-1)^i}{h^i} \int_0^1 F_{t,h,i,n}^{(i)}(\tau) y(t + (\tau - 1)h) d\tau$ , for  $n = 0, \dots, W$  and  $i = 0, \dots, N$ . Finally, this proof can be completed by substituting  $y$  by  $y^\varpi$  in (15).  $\square$

Error analysis for the estimation  $u_e$  given in Proposition 1 can be given thanks to previous studies. On one hand, according to (11), the estimation  $u_e$  contains a truncated term error (see [8, 9] for more theoretical analysis). On the other hand, according to (12), the truncated term error in (11) is a source of error for  $\tilde{\lambda}_{t,i}$ . Moreover,  $\tilde{\lambda}_{t,i}$  contain also a noise error contribution due to the noise in  $B_{y^\varpi}$  (see [9, 4] for similar theoretical analysis).

If we take a set of Jacobi orthogonal polynomials multiplied by their weight function and divided by the associated norms as the used modulating functions in Proposition 1, then the matrix  $A_f$  becomes the identity matrix.

Hence, the unknown coefficients can be directly given without solving the linear system (9). Thus, inspired by this idea, we can give the following corollary.

*Corollary 2.* Let  $y^\varpi$  be a noisy observation of the output  $y$  of the linear time-varying system defined by (1), and for  $n = 0, \dots, N$ ,

$$\forall \tau \in [0, 1], f_n(\tau) = w_{\alpha, \beta}(\tau) \frac{P_n^{(\alpha, \beta)}(\tau)}{\|P_n^{(\alpha, \beta)}\|_{\alpha, \beta}^2} \quad (16)$$

with  $\alpha, \beta \in ]M-1, +\infty[$ . Then, an estimate of the unknown input  $u$  in (1) can be given by:  $\forall t \in [h, T]$ ,

$$\forall \tau \in [0, 1], u_e(t + (\tau - 1)h) = \sum_{i=0}^N \tilde{\lambda}_{t,i} P_i^{(\alpha, \beta)}(\tau), \quad (17)$$

where  $\tilde{\lambda}_{t,i} = \sum_{i=0}^M \frac{(-1)^i}{h^i} \int_0^1 F_{t,h,i,n}^{(i)}(\tau) y^\varpi(t + (\tau - 1)h) d\tau$  with  $F_{t,h,i,n}(\tau) = f_n(\tau) a_i(t + (\tau - 1)h)$ .

**Proof.** Since  $w_{\alpha, \beta}(\tau) = (1 - \tau)^{\alpha} \tau^{\beta}$  with  $\alpha, \beta \in ]M-1, +\infty[$ , using the Leibniz formula we can verify that  $\{f_n\}_{n=0}^W$  is a set of  $(M-1)^{th}$  order modulating functions on  $[0, 1]$ . Then, according to the orthogonality of the Jacobi polynomial, we can deduce from (9) that:

$$\begin{aligned} A_f(n+1, i+1) &= \int_0^1 f_n(\tau) P_i^{(\alpha, \beta)}(\tau) d\tau \\ &= \begin{cases} 1, & \text{if } n = i, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

Consequently, this proof can be completed using (9).  $\square$

Another way to get Corollary 2 is to take the scalar product  $\langle \cdot, \cdot \rangle_{\alpha, \beta}$  involving the Jacobi polynomial  $\frac{P_n^{(\alpha, \beta)}(\cdot)}{\|P_n^{(\alpha, \beta)}\|_{\alpha, \beta}^2}$  to both sides of (10), such that the coefficients  $\lambda_{t,i}$  can be directly obtained. Then, after applying the generalized integration by parts, we substitute  $y$  by  $y^\varpi$  in the integrals involving  $y$ . Thus, the estimations of  $\lambda_{t,i}$  in Corollary 2 do not contain the error due to the truncated term in (11) any more.

Finally, if we fix the value of  $\tau$  in Proposition 1 and Corollary 2, we can get an estimated value of  $u$  on each sliding window  $[t-h, t]$ . Let us recall that when approximating a function by its truncated Jacobi series expansion on a sliding window, we usually have large truncated term errors near the two extremities of the sliding window (see, e.g., [20]). Consequently, according to [7, 8, 9], we can take the value of  $1 - \tau$  as the smallest root of  $P_{N+1}^{(\alpha, \beta)}$  to reduce the truncated term error. However, this choice of  $\tau$  produces a time-delay. More details are given in the numerical simulations section.

### 3.3 State estimation

In this subsection, we are going to estimate the successive derivatives  $y^{(i)}$ , for  $i = 0, \dots, M-1$ , using the estimated input  $u$  and the observation  $y^\varpi$  in a sliding integration window. For this purpose, we apply the extended modulating functions method. Unlike the classical modulating functions method where all the boundaries conditions are eliminated by the properties of the modulating functions,

we keep the right side boundary conditions which contain the derivative values of the output.

*Proposition 3.* Let  $y^\varpi$  be a noisy observation of the output  $y$  of the linear time-varying system defined by (1), and  $u_e$  be an estimation of the unknown input. Then, the successive derivatives of  $y$  can be estimated using a recursive way as follows:  $\forall t \in [h, T]$ ,

$$\begin{aligned} y_e(t) &= \frac{(-1)^{M-1}}{G_{t,h,M,0}^{(M-1)}(t)} \int_{t-h}^t g_{t,h,0}(\tau) u_e(\tau) d\tau \\ &+ \sum_{i=0}^M \frac{(-1)^{M+i}}{G_{t,h,M,0}^{(M-1)}(t)} \int_{t-h}^t G_{t,h,i,0}^{(i)}(\tau) y^\varpi(\tau) d\tau, \end{aligned} \quad (18)$$

and for  $n = 1, \dots, M-1$ ,

$$\begin{aligned} y_e^{(n)}(t) &= \frac{(-1)^{(M-n-1)}}{G_{t,h,M,n}^{(M-n-1)}(t)} \int_{t-h}^t g_{t,h,0}(\tau) u_e(\tau) d\tau \\ &+ \sum_{k=0}^{n-1} \sum_{i=M-n+k}^M \frac{(-1)^{i-k+M-n-1}}{G_{t,h,M,n}^{(M-n-1)}(t)} G_{t,h,i,n}^{(i-k-1)}(t) y_e^{(k)}(t) \\ &+ \sum_{i=0}^M \frac{(-1)^{i+M-n}}{G_{t,h,M,n}^{(M-n-1)}(t)} \int_{t-h}^t G_{t,h,i,0}^{(i)}(\tau) y^\varpi(\tau) d\tau, \end{aligned} \quad (19)$$

where  $g_{t,h,n}$ , for  $n = 0, \dots, M-1$ , is a  $(M-1, M-2-n)^{th}$  order modulating function on  $[t-h, t]$  with  $h \in ]0, T]$ ,  $G_{t,h,i,n}(\tau) = g_{t,h,n}(\tau) a_i(\tau)$ .

**Proof.**

**Step 1.** Application of integration by parts:

For any  $t \in [h, T]$  with  $h \in ]0, T]$ , we take a sequence of functions  $g_{t,h,n} \in \mathcal{C}^M([t-h, t])$ , for  $n = 0, \dots, M-1$ . Then, by multiplying both sides of (1) by  $g_{t,h,n}$  and integrating from  $t-h$  to  $t$ , we get:  $\forall t \in [h, T]$ ,

$$\begin{aligned} \int_{t-h}^t g_{t,h,n}(\tau) u(\tau) d\tau &= \\ \sum_{i=0}^M \int_{t-h}^t g_{t,h,n}(\tau) a_i(\tau) y^{(i)}(\tau) d\tau. \end{aligned} \quad (20)$$

Then, by applying the generalized integration by parts formula given in Lemma 1, we obtain: for  $i = 1, \dots, M$ , for  $n = 0, \dots, M-1$ ,  $\forall t \in [h, T]$ ,

$$\begin{aligned} \int_{t-h}^t G_{t,h,i,n}(\tau) y^{(i)}(\tau) d\tau &= \\ (-1)^i \int_{t-h}^t G_{t,h,i,n}^{(i)}(\tau) y(\tau) d\tau \\ &+ \sum_{k=0}^{i-1} (-1)^k \left[ G_{t,h,i,n}^{(k)}(\tau) y^{(i-1-k)}(\tau) \right]_{\tau=t-h}^{\tau=t}, \end{aligned} \quad (21)$$

where  $G_{t,h,i,n}(\tau) = g_{t,h,n}(\tau) a_i(\tau)$ .

**Step 2.** Elimination of the derivative values at  $t-h$ :

Let us assume that  $g_{t,h,n}$ , for  $n = 0, \dots, M-1$ , satisfies the property  $(P_2)$  by taking  $a = t-h$  and  $l = M-1$ . Then, we can deduce that  $G_{t,h,i,n}$  also satisfies the property  $(P_2)$  with  $a = t-h$  and  $l = i-1$ . Hence, all the derivative values at  $\tau = t-h$  in (21) are equal to 0. Thus, (21) becomes: for  $n = 0, \dots, M-1$ ,  $\forall t \in [h, T]$ ,

$$\begin{aligned}
& \sum_{i=0}^M \int_{t-h}^t G_{t,h,i,n}(\tau) y^{(i)}(\tau) d\tau = \\
& \sum_{i=0}^M (-1)^i \int_{t-h}^t G_{t,h,i,n}^{(i)}(\tau) y(\tau) d\tau \\
& + \sum_{i=1}^M \sum_{k=0}^{i-1} (-1)^k G_{t,h,i,n}^{(k)}(t) y^{(i-1-k)}(t).
\end{aligned} \quad (22)$$

Then, all the derivative values at  $t$  in (22) can be given in the following matrix:  $D =$

$$\begin{pmatrix} G_{t,h,1,n}^{(0)}(t) y^{(0)}(t) & \cdots & G_{t,h,M,n}^{(0)}(t) y^{(M-1)}(t) \\ 0 & \cdots & -G_{t,h,M,n}^{(1)}(t) y^{(M-2)}(t) \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (-1)^{M-1} G_{t,h,M,n}^{(M-1)}(t) y^{(0)}(t) \end{pmatrix}.$$

**Step 3.** Estimation of the derivative values at  $\tau = t$ :

We are going to calculate the boundary derivative values in the matrix  $D$  from the last line to the first line using extended modulating functions. For this purpose, we assume that  $g_n$ , for  $n = 0, \dots, M-1$ , also satisfies the property  $(P_3)$  with  $b = t$  and  $k = M-2-n$ . Hence,  $g_n$  is a  $(M-1, M-2-n)^{th}$  order modulating function on  $[t-h, t]$ . Moreover, we can deduce that  $G_{t,h,i,n}^{(k)}(t) = 0$ , for  $k = 0, \dots, \min(M-2-n, i-1)$ . Hence, the sum in (22) becomes:

$$\begin{aligned}
& \sum_{i=1}^M \sum_{k=0}^{i-1} (-1)^k G_{t,h,i,n}^{(k)}(t) y^{(i-1-k)}(t) = \\
& \sum_{i=M-n}^M \sum_{k=M-n-1}^{i-1} (-1)^k G_{t,h,i,n}^{(k)}(t) y^{(i-1-k)}(t).
\end{aligned} \quad (23)$$

By applying a change of index  $k \rightarrow i-1-k$  in (23), we get:

$$\begin{aligned}
& \sum_{i=M-n}^M \sum_{k=M-n-1}^{i-1} (-1)^k G_{t,h,i,n}^{(k)}(t) y^{(i-1-k)}(t) = \\
& \sum_{i=M-n}^M \sum_{k=0}^{i-M+n} (-1)^{(i-1-k)} G_{t,h,i,n}^{(i-1-k)}(t) y^{(k)}(t).
\end{aligned} \quad (24)$$

Then, by changing the order of the sums in (24), we get:

$$\begin{aligned}
& \sum_{i=M-n}^M \sum_{k=0}^{i-M+n} (-1)^{(i-1-k)} G_{t,h,i,n}^{(i-1-k)}(t) y^{(k)}(t) = \\
& \sum_{k=0}^n \sum_{i=M-n+k}^M (-1)^{(i-1-k)} G_{t,h,i,n}^{(i-1-k)}(t) y^{(k)}(t).
\end{aligned} \quad (25)$$

Hence, using (25), (20) becomes:  $\forall t \in [h, T]$ ,

$$\begin{aligned}
& \int_{t-h}^t g_{t,h,n}(\tau) u(\tau) d\tau = \\
& \sum_{i=0}^M (-1)^i \int_{t-h}^t G_{t,h,i,n}^{(i)}(\tau) y(\tau) d\tau \\
& + \sum_{k=0}^n \sum_{i=M-n+k}^M (-1)^{i-k-1} G_{t,h,i,n}^{(i-k-1)}(t) y^{(k)}(t),
\end{aligned} \quad (26)$$

for  $n = 0, \dots, M-1$ . By taking  $n = 0$  in (26) we get:  $\forall t \in [h, T]$ ,

$$\begin{aligned}
y(t) &= \frac{(-1)^{M-1}}{G_{t,h,M,0}^{(M-1)}(t)} \int_{t-h}^t g_{t,h,0}(\tau) u(\tau) d\tau \\
&+ \sum_{i=0}^M \frac{(-1)^{M+i}}{G_{t,h,M,0}^{(M-1)}(t)} \int_{t-h}^t G_{t,h,i,0}^{(i)}(\tau) y(\tau) d\tau.
\end{aligned} \quad (27)$$

Then, for any  $n \in \{1, \dots, M-1\}$ , we get:  $\forall t \in [h, T]$ ,

$$\begin{aligned}
y^{(n)}(t) &= \frac{(-1)^{(M-n-1)}}{G_{t,h,M,n}^{(M-n-1)}(t)} \int_{t-h}^t g_{t,h,0}(\tau) u(\tau) d\tau \\
&+ \sum_{k=0}^{n-1} \sum_{i=M-n+k}^M \frac{(-1)^{i-k+M-n-1}}{G_{t,h,M,n}^{(M-n-1)}(t)} G_{t,h,i,n}^{(i-k-1)}(t) y^{(k)}(t) \\
&+ \sum_{i=0}^M \frac{(-1)^{i+M-n}}{G_{t,h,M,n}^{(M-n-1)}(t)} \int_{t-h}^t G_{t,h,i,0}^{(i)}(\tau) y(\tau) d\tau.
\end{aligned} \quad (28)$$

Finally, this proof can be completed by substituting  $y$  by  $y^\varpi$  and  $u$  by  $u_e$  in (27) and (28). Thus, the derivatives  $y^{(n)}$ , for  $n = 1, \dots, M-1$ , can be estimated in a recursive way with the estimates of  $y^{(k)}$ , for  $k = 0, \dots, n-1$ .  $\square$

Finally, we can remark that except the estimation error in the estimation of  $u$ , the differentiators proposed in Proposition 3 contain only the noise error contribution (see [4] for similar theoretical analysis).

#### 4. SIMULATION RESULTS

In order to illustrate the accuracy and robustness against corrupting noises of the proposed estimators, we present some numerical results in this section.

Let us consider the following simplified model of a DC motor system (electric part is neglected) [15]:

$$\forall t \in I, \begin{cases} \dot{x}_1 = x_2(t), \\ \dot{x}_2 = -\frac{1}{a(t)} x_2(t) + \frac{k}{a(t)} u(t), \end{cases} \quad (29)$$

where  $x_1$  is the angular position of the rotor,  $x_2$  is the angular velocity of the rotor,  $y \equiv x_1$  is the output and  $u$  is the input. The parameter  $k$  is a strictly positive constant, and  $a$  is a time-varying strictly positive parameter. According to (29), we can obtain the following linear differential equation:

$$\forall t \in I, \frac{a(t)}{k} \ddot{y}(t) + \frac{1}{k} \dot{y}(t) = u(t). \quad (30)$$

From now on, we assume that  $y^\varpi(t_i) = y(t_i) + \delta\varpi(t_i)$  is a discrete noisy observation of the output on  $I = [0, 20]$ , with  $t_i = T_s i$ , for  $i = 0, \dots, m$ , where  $T_s = \frac{20}{m} = 10^{-4}$  ( $m = 2 \times 10^5$ ) is an equidistant sampling period.  $\delta\varpi(t_i)$  is simulated from a zero-mean white Gaussian *iid* sequence, where the variance  $\delta^2$  is adjusted such that the signal-to-noise ratio  $\text{SNR} = 10 \log_{10} \left( \frac{\sum |y^\varpi(t_i)|^2}{\sum |\delta\varpi(t_i)|_2^2} \right)$  is equal to  $\text{SNR} = 25\text{dB}$ .

Moreover, we assume that the initial conditions are such that  $x_1(0) = 1$  and  $x_2(0) = 0$ , the input is a sinusoidal function  $u(t_i) = 12 \sin(t_i)$ , the parameters  $k = 1$  and  $a(t_i) = 2 \cos(0.2\pi t_i) + 3$ . We can see the output  $y$  and its discrete noisy observation  $y^\varpi$  in Figure 1.

Firstly, we estimate  $y \equiv x_1$  and  $\dot{y} \equiv x_2$  using Proposition 3 in the case where we assume that the input  $u$  is known. In this example, we have  $M = 2$ . Hence, according to Proposition 3, we need two modulating functions of

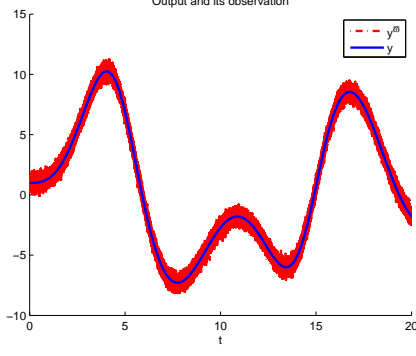


Fig. 1. The noise-free output and its noisy observation.

$(1,0)^{th}$  order and  $(1,-1)^{th}$  order respectively. For this purpose, we take  $g_{t_i,h,0}(\tau) = (t_i - \tau)(\tau - t_i + h)^2$  and  $g_{t_i,h,1}(\tau) = (\tau - t_i + h)^2$  where  $\tau \in [t_i - h, t_i]$  with  $t_i \geq h$  and  $h = 1.25$ . This kind of modulating functions that have a similar form to the weight function of the Jacobi polynomial have been already used in [4, 26]. Moreover, we apply the trapezoidal numerical integration method to approximate the integrals obtained in Proposition 3. We can see the obtained estimations and the associated absolute estimation errors in Figure 2.

Secondly, we estimate both the unknown input  $u$  and the state variables. On one hand, we use Corollary 2 to estimate  $u$  by taking  $h = 1.25$ ,  $N = 1$ , and  $\alpha = \beta = 3$  such that  $f_n$  is a  $1^{st}$  order modulating function on  $[0, 1]$ , for  $n = 0, 1, 2$ . Moreover, as shown in Subsection 3.2, we take  $1 - \tau = 0.3333$  as the smallest root of  $P_2^{(\alpha,\beta)}$  to reduce the truncated term error. Hence, this choice of  $\tau$  produces a time-delay in the estimation of  $u$ . On the other hand, we take the same modulating functions as before and the obtained estimation of  $u$  in Proposition 3. Since the estimation of  $u$  contains a time-delay of value  $h(1 - \tau)$ , it produces also a time-delay with the same value in the estimation of  $y$  and  $\dot{y}$ . We can see the obtained estimations, the shifted estimations and the associated absolute estimation errors in Figure 3.

## 5. CONCLUSIONS

In this paper, we have extended the classical modulating functions method to the numerical differentiation problem so as to estimate the states of a class of linear time-varying systems with unknown inputs. The design of the input and the state estimators has been inspired by the algebraic model-free differentiators and the algebraic model-based differentiators respectively, without using a dynamic auxiliary system. Hence, these estimators have similar advantages to the recent algebraic estimation method. Indeed, they have been easily obtained without knowing the statistical properties of noises and have been exactly given by integral formulae leading to non-asymptotic properties and robustness against corrupting noises.

Different from the model-free differentiators obtained in [7, 8] using the noisy measurement of the signal to differentiate, the proposed input estimators were obtained via the differential equation defining the system. Moreover, unlike the model-based differentiator obtained in [15, 16] with complex mathematical deduction, the proposed state

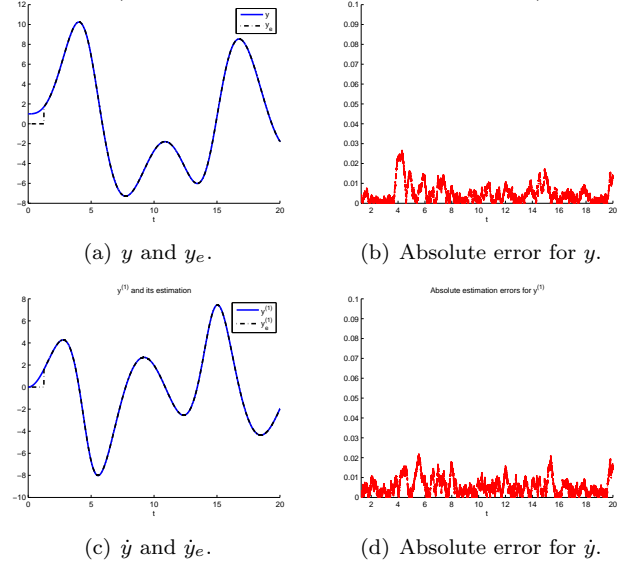


Fig. 2. State estimations where the input  $u$  is known.

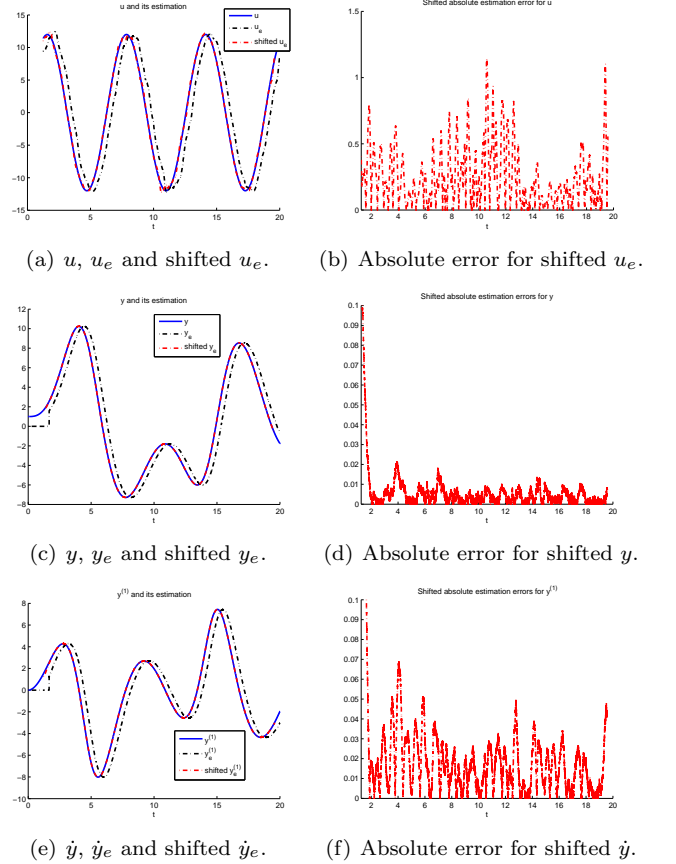


Fig. 3. State estimations where the input  $u$  is unknown.

estimators are much easier to obtain and to understand thanks to the use of the generalized integration by parts formula and extended modulating functions.

Numerical results have been given in both cases where the input is known or not. In the first case, the state has been accurately estimated without time-delay. In the second case, a time-delay with a known value has been introduced to improve the estimation of the unknown input, which

has produced a time-delay with the same value in the estimations of the state. This time-delay was due to the truncated term error in the estimation of the input, and its value has been taken using the root of the first term in the truncated terms. In our future work, we are interested in the extension of the proposed method to state estimation for non-linear systems.

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